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Boundedness of Solutions for Periodic Linear Differential Equations

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1 Introduction

Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{R}$ the real line.

The purpose of the present paper is to characterize, by the sets of initial values, the boundedness of solutions and τ-periodic solutions for the periodic linear differential equation of the form

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t), \tag{1}$$

where $A(t)$ is a τ-periodic continuous $p\times p$ matrix function with period $\tau > 0$ and $f : \mathbb{R} \to \mathbb{C}^p$ a τ-periodic continuous function. It is related with a new representation of solutions of the linear difference equation of the form

$$x_{n+1} = Bx_n + b, \quad x_0 = w, \tag{2}$$

where B is a complex $p \times p$ matrix and $b \in \mathbb{C}^p$.

More recently, Kato, Naito and Shin [4] gave a new representation of solutions of a special case of the equation (2), that is, the linear difference equation of the form

$$x_{n+1} = e^{\tau A}x_n + b, \quad x_0 = w, \tag{3}$$

where A is a complex $p\times p$ matrix and $\tau > 0$. Using the representation of solutions, we obtained a new representation of solutions and the complete classification of the sets of initial values according to the asymptotic behavior of solutions for the case where $A(t) = A$ in the equation (1).

This paper is based on the idea of the paper [4] and the characteristis multiplier on a periodic operator.

2 Linear difference equations

2.1 A representation of solutions of difference equations

Throughout this paper we use the following notations: Let E be the unit $p \times p$ matrix. For a complex $p \times p$ matrix H we denote by $\sigma(H)$ the set of all eigenvalues of H, and by $h_H(\eta)$ the index of $\eta \in \sigma(H)$. Let $M_H(\eta) = \mathcal{N}((H - \eta E)^{h_H(\eta)})$ be the generalized eigenspace corresponding to $\eta \in \sigma(H)$. Let $Q_\eta(H) : \mathbb{C}^p \to M_H(\eta)$ be the projection corresponding to the direct sum decomposition

$$\mathbb{C}^p = \sum_{\eta \in \sigma(H)} \oplus M_H(\eta).$$

These projections have the following properties:

$$Q_\eta(H)\mathbb{C}^p = M_H(\eta), \quad HQ_\eta(H) = Q_\eta(H)H,$$

$$Q_\eta(H)Q_\zeta(H) = 0 \ (\eta \neq \zeta), \quad Q_\eta(H)^2 = Q_\eta(H), \quad E = \sum_{\eta \in \sigma(H)} Q_\eta(H).$$

The solution $\{x_n\}$ of the equation(2) is given as

$$x_n := x_n(w, b) = B^n w + S_n(B)b,$$

where

$$S_n(B) = \sum_{k=0}^{n-1} B^k, \quad (n \geq 1), \quad S_0(B) = 0.$$

Let $h(\mu) = h_B(\mu)$, $Q_\mu = Q_\mu(B)$ for $\mu \in \sigma(B)$. Then

$$Q_\mu x_n(w, b) = B^n Q_\mu w + S_n(B) Q_\mu b. \tag{4}$$

In this section, we will rearrange the right side of this representation by collecting the terms which are the same order with respect to n.

To describe the results, we prepare the following notations. For any $\mu \in \sigma(B)$ such that $\mu \neq 1$, we define a matrix $Z_\mu(B)$ as follows:

$$Z_\mu(B) = Z_\mu(B, h(\mu))$$

where

$$Z_\mu(B, h) = -\sum_{k=0}^{h-1} \frac{1}{(1-\mu)^{k+1}} (B - \mu E)^k, \quad (\mu \neq 1),$$

for $h = 1, 2, \cdots, h(\mu)$. Furthermore, we set

$$\gamma_B(Q_\mu w, Q_\mu b) = Q_\mu w + Z_\mu(B) Q_\mu b \quad (\mu \neq 1),$$

$$\delta_B(Q_\mu w, Q_\mu b) = (B - E)Q_\mu w + Q_\mu b \quad (\mu = 1).$$

We use the well known notation $(n)_k$ such that

$$(n)_k = \begin{cases} 1, & (k = 0), \\ n(n-1)(n-2)\cdots(n-k+1), & (k = 1, 2, \cdots, n), \\ 0, & (k = n+1, n+2, \cdots). \end{cases}$$

Put

$$B_{k,\mu} = \frac{1}{k!\mu^k}(B - \mu E)^k \quad (\mu \neq 0, \quad \mu \in \sigma(B)). \tag{5}$$

The following result is a key in this paper.

Theorem 2.1 *Let B be non-singular and $\mu \in \sigma(B)$. The component $Q_\mu x_n(w, b)$ of the solution $x_n(w, b)$ of the equation (2) is expressed as follows:*

1) *If $\mu \neq 1$, then*

$$\begin{aligned} Q_\mu x_n(w, b) &= \mu^n \sum_{k=0}^{h(\mu)-1} (n)_k B_{k,\mu} \gamma_B(Q_\mu w, Q_\mu b) - Z_\mu(B) Q_\mu b \\ &= B^n \gamma_B(Q_\mu w, Q_\mu b) - Z_\mu(B) Q_\mu b. \end{aligned}$$

2) *If $\mu = 1$, then*

$$Q_\mu x_n(w, b) = \sum_{k=0}^{h(\mu)-1} \frac{1}{k+1}(n)_{k+1} B_{k,\mu} \delta_B(Q_\mu w, Q_\mu b) + Q_\mu w.$$

2.2 Bounded solutions and constant solutions

In this section, the boundedness of solutions and constant solutions to the equation (2) are characterized by using representations of solutions obtained in the previous sections. The following results on the boundedness of solutions follows from Theorem 2.1 immediately.

Theorem 2.2

I *The solution $x_n(w, b)$ of the equation (2) is bounded if and only if the following conditions hold: For every $\mu \in \sigma(B)$,*

(1) *if $|\mu| > 1$, then $\gamma_B(Q_\mu w, Q_\mu b) = 0$;*

(2) *if $\mu \neq 1$ and $|\mu| = 1$, then $(B - \mu E)\gamma_B(Q_\mu w, Q_\mu b) = 0$.*

(3) *if $\mu = 1$, then $\delta_B(Q_\mu w, Q_\mu b) = 0$;*

II *The following statements are equivalent:*

1) *The solution x_n of the equation (2) with $x_0 = w$ is constant.*

2) *For every* $\mu \in \sigma(B)$, *the following conditions hold:*
(1) *if* $\mu \neq 1$, *then* $\gamma_B(Q_\mu w, Q_\mu b) = 0$;
(2) *if* $\mu = 1$, *then* $\delta_B(Q_\mu w, Q_\mu b) = 0$.
3)
$$(E - B)w = b.$$

Using the same argument as in the proof of Lemma 5.6 in [4], we can obtain the necessary and sufficient conditions on the existence of bounded solution for the equation (2).

Theorem 2.3 *The following statements are equivalent:*
1) *The equation* (2) *has a solution which is bounded;*
2) *There is a* $Q_\mu w$ *such that* $\delta_B(Q_\mu w, Q_\mu b) = 0$ *is satisfied ;*
3) *If* $\mu = 1 \in \sigma(B)$, *then* $Q_\mu b \in (B - \mu E)M_B(\mu)$;
4) *If* $\mu = 1 \in \sigma(B)$, *then* $b \in \mathcal{R}(B - \mu E)$, *the range of* $B - \mu E$.

Corollary 2.1 *All bounded solutions* $x_n(w, b)$ *of the equation* (2) *are constant solutions whenever* $\gamma_B(Q_\mu w, Q_\mu b) = 0$ *in the case where* $|\mu| \leq 1, \mu \neq 1$.

Corollary 2.2 *Assume that* $\mu \neq 1, \mu \in \sigma(B)$.
1) *There are a bounded solution and a constant solution to the equation* (2).
2) *A bounded solution* $x_n(w, b)$ *of the equation* (2) *is a constant solution if and only if* $\gamma_B(Q_\mu w, Q_\mu b) = 0$ *for all* $\mu \in \sigma(B)$.

3 Bounded solutions of periodic linear differential equations

In this section, we give criteria on the existence of bounded solutions on $\mathbb{R}_+$ and τ-periodic solutions to the equation (1); that is,

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t),$$

where $A(t)$ is a τ-periodic continuous $p \times p$ matrix function with period $\tau > 0$ and $f : \mathbb{R} \to \mathbb{C}^p$ a τ-periodic continuous function. We will use two methods: the first method is based on the characteristic multiplier; the second one is based on the characteristic exponent.

3.1 Periodic maps

Now we state the properties of the solution operators $U(t,s)$ of the homogeneous equation corresponding to the equation (1). The operator $U(t,s)$ is defined as

$$U(t,s)w = u(t;s,w) \quad w \in \mathbb{C}^p$$

by using the unique solution $u(t;s,w)$ of the equation $u'(t) = A(t)u(t)$ with the initial condition $u(s) = w \in \mathbb{C}^p$.

Lemma 3.1 *The solution operators $U(t,s), (t,s \in \mathbb{R})$, have the following properties:*
1) $U(t,t) = E$ *for all* $t \in \mathbb{R}$.
2) $U(t,s)U(s,r) = U(t,r)$.
3) *The map* $(t,s,x) \mapsto U(t,s)x$ *is continuous for* $(t,s,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{C}^p$.
4) $U(t+\tau, s+\tau) = U(t,s)$.
5) $U^n(s+\tau,s) = U(s+n\tau,s)$.
6) $U(t+n\tau,s) = U^n(t+\tau,t)U(t,s) = U(t,s)U^n(s+\tau,s)$.
7) $U(t,s)$ *is a nonsingular matrix and* $U(t,s)^{-1} = U(s,t)$.

Since $U(\tau,0)$ is a nonsingular matrix, we can take a matrix A such that

$$U(\tau,0) = e^{\tau A}.$$

Define

$$P(t) = U(t,0)e^{-tA}.$$

Then it is easy to see that $P(t+\tau) = P(t)$. We have thus the representation by Floquet:

$$U(t,0) = P(t)e^{tA}.$$

It is easy to see that

$$U(t,s) = P(t)e^{(t-s)A}P^{-1}(s). \tag{6}$$

Since $U(t,0)^{-1}$ exists, we have

$$P^{-1}(t) = e^{tA}U(t,0)^{-1}.$$

Moreover, since $P(t)$ is τ-periodic, $P^{-1}(t)$ is also τ-periodic; clearly $P(\tau) = P(0) = E$, $P^{-1}(\tau) = P^{-1}(0) = E$.

Define the well known periodic map (operator) (or the Poincareé map or the monodromy operator) $V(t), t \in \mathbb{R}$ by

$$V(t) = U(t,t-\tau) = U(t+\tau,t).$$

Then $V(0) = U(\tau,0) = e^{\tau A}$, and it is easy to check the following properties.

$$V(t+\tau) = V(t), \quad V(t)U(t,s) = U(t,s)V(s).$$

It follows from the relation (6) that

$$V(t) = P(t)V(0)P(t)^{-1}. \tag{7}$$

Now we recall elementary results in linear algebra. Let C and D be square matrices with the same size. Assume that there exits a nonsingular matrix T such that $TC = DT$. Then the following properties hold true.

a) $\sigma(C) = \sigma(D)$.

b) For $\gamma \in \sigma(C)$, $TQ_\gamma(C) = Q_\gamma(D)T$.

We now will return to the equation (1). Set

$$Q_\mu(t) = Q_\mu(V(t)), \ (\mu \in \sigma(V(t))).$$

We prepare a well known lemma, cf. [2].

Lemma 3.2 *For $t, s \in \mathbb{R}$ the following relations hold:*

1) $\sigma(V(t)) = \sigma(V(s)) = \sigma(V(0)), \quad t, s \in \mathbb{R}.$

2)

$$Q_\mu(t)U(t,s) = U(t,s)Q_\mu(s) \ \text{ for } \mu \in \sigma(V(0)).$$

3) *Let $\mu \in \sigma(V(0))$. Then $h_{V(s)}(\mu) = h_{V(t)}(\mu)$ and*

$$U(t,s)M_{V(s)}(\mu) = M_{V(t)}(\mu).$$

For a $\mu \in \sigma(V(0))$, where $V(0) = e^{\tau A}$ as described before, we set

$$\sigma_\mu(A) = \{\lambda \in \sigma(A) \mid \mu = e^{\tau\lambda}\}.$$

Lemma 3.3 *The following results hold true:*

1)

$$Q_\mu(t) = P(t)Q_\mu(0)P^{-1}(t) = \sum_{\lambda \in \sigma_\mu(A)} P(t)P_\lambda P^{-1}(t).$$

2)

$$M_{V(t)}(\mu) = P(t)M_{V(0)}(\mu) = P(t) \sum_{\lambda \in \sigma_\mu(A)} \oplus M_A(\lambda).$$

Proof From (7), $V(t)P(t) = P(t)V(0)$ holds. This implies that

$$Q_\mu(t)P(t) = P(t)Q_\mu(0)$$

and that

$$M_{V(t)}(\mu) = P(t)M_{V(0)}(\mu).$$

On the other hand, since

$$M_{V(0)}(\mu) = \sum_{\lambda \in \sigma_\mu(A)} \oplus M_A(\lambda), \tag{8}$$

we have

$$Q_\mu(0) = \sum_{\lambda \in \sigma_\mu(A)} P_\lambda,$$

from which the remainder follows. □

Remark 3.4 *From* 3) *in Lemma* 3.2 *and* 2) *in Lemma* 3.3 *we note that*

$$M_{V(t)}(\mu) = U(t,0)M_{V(0)}(\mu) = P(t)M_{V(0)}(\mu), \quad (t \in \mathbb{R}).$$

3.2 Bounded solutions and τ-periodic solutions

We consider general criteria on the existence of bounded solutions and τ-periodic solutions for the equation (1) by using characteristic multipliers.

Now, we reduce the equation (1) to a difference equation as follows. Let $x(t) := x(t;0,w)$ be the solution of the equation (1) such that $x(0) = w$, For any $t \in [0,\infty)$ there is an $n \in \mathbb{N} \cup \{0\}$ such that $0 \leq t - n\tau < \tau$. Then

$$x(t) = U(t,n\tau)x(n\tau) + \int_{n\tau}^{t} U(t,s)f(s)ds, \quad n \in \mathbb{N} \cup \{0\}. \tag{9}$$

Setting $x_n = x(n\tau)$, (9) is reduced to the difference equation of the form

$$x_{n+1} = U(\tau,0)x_n + b_f, \quad x_0 = w. \tag{10}$$

Denote by $x_n(w,b_f)$ the solution of the equation (10). Then (9) is expressed as

$$x(t) = U(t,n\tau)x_n(w,b_f) + \int_{n\tau}^{t} U(t,s)f(s)ds.$$

By using the relation $Q_\mu(n\tau) = Q_\mu(\tau) = Q_\mu(0)$ and Lemma 3.2, we have

$$Q_\mu(t)x(t) = U(t,n\tau)Q_\mu(0)x_n(w,b_f) + \int_{n\tau}^{t} U(t,s)Q_\mu(s)f(s)ds.$$

It is obvious that $x(t)$ is bounded on $\mathbb{R}_+$ if and only if $Q_\mu(t)x(t)$ is bounded on $\mathbb{R}_+$ for every $\mu \in \sigma(V(0))$. Since

$$\sup_{0 \leq t-s \leq \tau} \|U(t,s)\| < \infty,$$

it follows that

(1) $Q_\mu(t)x(t)$ is bounded on $\mathbb{R}_+$ if and only if $\{Q_\mu(0)x_n(w,b_f)\}$ is bounded; and

(2) $Q_\mu(t)x(t)$ is τ-periodic if and only if $\{Q_\mu(0)x_n(w,b_f)\}$ is constant.

Using these facts and Theorem 2.1 with (10), the following result is easily obtained.

Theorem 3.1 *The following statements hold true.*

1) *The solution of the equation* (1) *with* $x(0) = w$ *is bounded on* $\mathbb{R}_+$ *if and only if the following conditions hold: For every* $\mu \in \sigma(V(0))$,

(1) *if* $|\mu| > 1$, *then* $\gamma_{V(0)}(Q_\mu(0)w, Q_\mu(0)b_f) = 0$;

(2) *if* $\mu \neq 1$ *and* $|\mu| = 1$, *then* $(V(0) - \mu E)\gamma_{V(0)}(Q_\mu(0)w, Q_\mu(0)b_f) = 0$.

(3) *if* $\mu = 1$, *then* $\delta_{V(0)}(Q_\mu(0)w, Q_\mu(0)b_f) = 0$;

2) *The solution of the equation* (1) *is* τ*-periodic if and only if for every* $\mu \in \sigma(V(0))$, *the following conditions hold:*

(1) *if* $\mu \neq 1$, *then* $\gamma_{V(0)}(Q_\mu(0)w, Q_\mu(0)b_f) = 0$;

(2) *if* $\mu = 1$, *then* $\delta_{V(0)}(Q_\mu(0)w, Q_\mu(0)b_f) = 0$.

Needless to say, we can easily obtain the results corresponding to Theorem 2.3, Corollary 2.1 and 2.2.

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